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USE OF THE ENERGY CRITERION OF FRACTURE TO DETERMINE THE SHAPE OF A SLIGHTLY CURVED CRACK

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An asymptotic formula is obtained for the total-energy increment during quasistatic growth of a semi-infinite crack in an anisotropic elastic plane under complex loading. It is assumed that the shear loads are much larger than the tearing loads. The shape of the slightly curved crack was determined using the Griffith criterion in two versions: global and local. It is shown, in particular, that the first version leads to an improbable result.

Key words: *slightly curved crack, local and global Griffith fracture criteria, asymptotic analysis.*

1. Energy Criterion of Fracture. The Griffith criterion is usually formulated as follows: During quasistatic propagation, the crack chooses the path that ensures the lowest total energy (potential and surface) at any time. In this connection, two questions arise: 1) What time is considered if the fracture process is quasistatic; 2) How should the minimization problem be formulated: globally, for the entire time interval, or locally, for each time?

The answer to the first question is simple enough: The loading parameter τ is required that corresponds to a slower process than wave propagation in a solid and increases monotonically together with real time t. In principle, the parameter τ can differ in dimension from t but its introduction allows one to ignore the inertial terms in the equilibrium equations [1].

The answer to the second question is not so obvious since for a complex stress state, the problem of calculating the total energy as a function of the crack shape using the modern mathematical apparatus cannot be solved. In the present study, the asymptotic energy was found using the simplifying assumption that the tearing load far exceeds the shear load. As a consequence, the crack path remains close to a straight line and its slight curvature is taken into account using asymptotic methods.

In the present work, we studied the elongation and curving of a semi-infinite crack which initially passes along the axis of elastic and strength symmetry of an anisotropic plane. Approximate but asymptotically accurate formulas were obtained and used to show that the global formulation of the Griffith criterion (minimization of the total energy on the entire interval $[0, \tau]$) leads to a paradoxical result: the crack branch is straight-line and the angle of its deviation is determined only by the load at the time τ but not by the loading prehistory. This conclusion is inconsistent with the assumption on the quasistatic nature of the crack growth because if the loading is not simple, then, at any time $\tau' \in (0, \tau)$, the crack deviation from the initial axis is different. The reason for the inadequacy of the global formulation is apparently the absence of the potentiality property of the total energy functional since the fracture is irreversible. It is shown that the local formulation of the Griffith criterion (minimization of the total energy increment on elementary intervals) is free from the disadvantages mentioned above.

2. Formulation of the Crack Problem. Let a semi-infinite crack in a homogeneous elastic plane $\mathbb{R}^2$ be defined by the formula

$$\Lambda_l := \Lambda_l(h, H) = \{x = (x_1, x_2)\colon x_1 \le l,\ x_2 = hH(x_1)\}. \tag{2.1}$$

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The origin of the Cartesian coordinates x coincides with the initial position of the tip O of a straight crack, i.e., $H(x_1) = 0$ at $x_1 \leq 0$; H is a continuous function which is smooth at $x_1 \geq 0$, but its derivatives can undergo discontinuities of the first kind at the point $x_1 = 0$. The quantity $l \geq 0$ is the crack length increment, and the function H and the small dimensionless parameter $h > 0$ describe the slight curvature of the crack path.

The plane with the crack is loaded at infinity and the displacement vector $u^l = (u_1^l, u_2^l)$ satisfies the homogeneous equilibrium equations and the boundary conditions

$$-\frac{\partial}{\partial x_1}\sigma_{1k}(u^l;x) - \frac{\partial}{\partial x_2}\sigma_{2k}(u^l;x) = 0 \qquad (k=1,2, \quad x \in \mathbb{R}^2 \setminus \Lambda_l); \tag{2.2}$$

$$n_1^{\pm}(x)\sigma_{1k}(u^l;x) + n_2^{\pm}(x)\sigma_{2k}(u^l;x) = 0 \qquad (k=1,2, \quad x \in \Lambda_l^{\pm}), \tag{2.3}$$

and the asymptotic condition

$$u^l(x) = C_1(l)X^1(x) + C_2(l)X^2(x) + O(|x|^{-1/2}), \qquad |x| \to \infty. \tag{2.4}$$

Here $\sigma_{jk}(u)$ are the Cartesian components of the stress tensor generated by the displacement vector u; $n^{\pm} = (n_1^{\pm}, n_2^{\pm})$ is the unit outward normal vector on the faces $\Lambda_l^{\pm}$ of the crack Λ_l. The quantities $C_1(l)$ and $C_2(l)$, which have the dimension of the stress intensity factor (SIF), describe the external actions and are defined so that for each $l \geq 0$, the crack Λ_l is critical (for the definition see Sec. 5). The symbols X^1 and X^2 are used to denote the displacement fields that generate square-root singularities of the stresses and, hence, satisfy the ordinary normalization conditions on the crack extension $\Lambda_0 = \{x\colon\ x_1 \leq 0, x_2 = 0\}$:

$$\sigma_{12}(X^j;x_1,0) = (2\pi x_1)^{1/2}\delta_{j2}, \quad \sigma_{22}(X^j;x_1,0) = (2\pi x_1)^{1/2}\delta_{j1}, \quad j=1,2, \quad x_1 > 0. \tag{2.5}$$

Here δ_{jk} is the Kronecker symbol. In addition, the smooth functions $l \mapsto C_i(l)$ are assumed to satisfy the relations

$$C_1(l) > 0; \tag{2.6}$$

$$C_2(l) = h\overline{C}_2(l), \tag{2.7}$$

where $|\overline{C}_2(l)| \leq C_1(l)$ at all $l \geq 0$; $h > 0$ is a small parameter [see (2.1)]. We assume that the abscissa Ox_1 passes through the plane of symmetry of the physical properties of the solid. As is shown in [2], inequality (2.6) ensures the absence of contact of the crack faces (for arbitrary anisotropy, the latter, generally speaking, is incorrect).

Since outside a certain circle, the crack Λ_l is straight-line, formula (2.4) can be specified by using lower-order asymptotic terms $O(|x|^{-1/2})$:

$$u^l(x) = \sum_{j=1}^{2}(C_j(l)X^j(x) + N_j(l)Y^j(x)) + O(|x|^{-1}), \qquad |x| \to \infty. \tag{2.8}$$

Here $N_1(l)$ and $N_2(l)$ are coefficients that depend on the SIF $C_j(l)$, the elastic properties of the material, and the crack shape (2.1). The vector functions Y^j, which are positively homogeneous and have a power of $-1/2$, obey the biorthogonality conditions (see, for example, [2, 4, 5])

$$q(X^j, Y^k; \Gamma) := \int\limits_{\Gamma} (Y^k(x)\sigma^{(n)}(X^j;x) - X^j(x)\sigma^{(n)}(Y^k;x))\, ds_x = \delta_{jk} \quad (j=1,2,\ k=1,2), \tag{2.9}$$

where Γ is a simple smooth arch which connects the faces of the crack Λ_0 and encompasses its tip and $\sigma^{(n)} = (\sigma_1^{(n)}, \sigma_2^{(n)})$ [$\sigma_j^{(n)} = n_1\sigma_{1j} + n_2\sigma_{2j}$, where $n = (n_1, n_2)$ is the unit outward normal vector for the region Ω bounded by Γ]. On the right side of (2.9), the integral is invariant and, by virtue of the symmetry of the elastic properties, the following equality is satisfied: [2]

$$\frac{\partial X^j}{\partial x_1}(x) = -m_{jj}Y^j(x) \qquad (j=1,2); \tag{2.10}$$

here $m_{11} > 0$ and $m_{22} > 0$ are material constants. In the isotropic case, $m_{11} = m_{22} = \mu^{-1}(1-\nu)$, where $\mu > 0$ is the shear modulus and $0 \leqslant \nu < 1/2$ is the Poisson factor.

Relation (2.10) contains the derivatives of the fields X^j along the straight-line crack Λ_0, which are proportional to the weight functions [6] or the dual singular solutions Y^j [4]. In [2], the following relation between Y^2 and the derivative $\partial X^1/\partial x_2$ across the crack was obtained:

$$\frac{\partial X^1}{\partial x_2}(x) = -\frac{m_{11}}{m_{22}}\,\frac{\partial X^2}{\partial x_1}(x) = m_{11}Y^2(x). \tag{2.11}$$

Formula (2.11) differs from formula (2.6) in [2] because (2.11) uses the basis $\{X^1, X^2\}$ adapted to the force fracture criteria whereas formula (2.6) in [2] uses a basis adapted to the strain fracture criteria; these bases are related by formulas (1.8) and (1.7) in [2].

3. Energy Functional. Since the solution u^l of the problem (2.2)–(2.4) in the unbounded region $\mathbb{R}^2 \setminus \Lambda_l$ increases as $O(|x|^{1/2})$ for $|x| \to \infty$, the integral of the elastic energy

$$E(u^l; \mathbb{R}^2 \setminus \Lambda_l) = \frac{1}{2}\sum_{j,k=1}^{2}\int\limits_{\mathbb{R}^2\setminus\Lambda_l} \sigma_{jk}(u^l; x)\,\frac{\partial u^l_j}{\partial x_k}(x)\,dx$$

diverges. To impart meaning to the energy functional, we interpret the solution u^l as the limit of solutions of the problems for large but bounded bodies:

$$-\frac{\partial}{\partial x_1}\,\sigma_{1k}(u^l_{(R)}; x) - \frac{\partial}{\partial x_2}\,\sigma_{2k}(u^l_{(R)}; x) = 0, \qquad k = 1, 2, \quad x \in \Omega_R \setminus \Lambda_l; \tag{3.1}$$

$$n_1^{\pm}(x)\sigma_{1k}(u^l_{(R)}; x) + n_2^{\pm}(x)\sigma_{2k}(u^l_{(R)}; x) = 0, \qquad k = 1, 2, \quad x \in \Lambda_l^{\pm} \cap \Omega_R; \tag{3.2}$$

$$\sigma^{(n)}(u^l_{(R)}; x) = g^l_{(R)}(x) := \sum_{j=1}^{2} C_{1j}(l)\sigma^{(n)}(X^j; x), \qquad x \in \partial\Omega_R. \tag{3.3}$$

Here $\Omega_R = \{x\colon\ R^{-1}x \in \Omega\}$ is the R-fold extension of a certain region $\Omega \subset \mathbb{R}^2$ which contains the point O. Considering the ratio l/R a small parameter, we seek an asymptotic representation of solutions of problem (3.1)–(3.3) in the form

$$u^l_{(R)}(x) = u^l(x) + R^{-1/2}v(R^{-1}x) + \tilde{u}^l_{(R)}(x). \tag{3.4}$$

Using the method of composite expansions [7] (see also [3, 8–10] for elasticity problems), we conclude that the main term of the asymptotic relation (3.4) is a solution of the problem (2.2)–(2.4) and the correction v is a bounded solution for a solid Ω with an edge crack $\Lambda_0 \cap \Omega$ (without a branch) that compensates for the residual of the field u^l in the boundary condition (3.3). Thus, according to expansion (2.8), we have

$$\sigma^{(n)}(v; \xi) = -N_1(l)\,\sigma^{(n)}(Y^1; \xi) - N_2(l)\,\sigma^{(n)}(Y^2; \xi), \qquad \xi \in \partial\Omega. \tag{3.5}$$

Finally, $\tilde{u}^l_{(R)}$ is a small residue, which, outside the tip of the crack Λ_l, under a suitable normalization eliminating arbitrariness in the choice of rigid displacement, obeys the inequality [7, 9]

$$|\tilde{u}^l_{(R)}(x)| \le c m_{11} C_1(l)\,\frac{l}{R}\,(l + |x|)^{-1/2} \tag{3.6}$$

with a constant c that does not depend on the geometrical parameters R and l.

By virtue of the Clapeyron theorem and formulas (3.4) and (3.6), the potential strain energy (elastic energy ignoring the work of external forces) stored in the solid $\Omega_R \setminus \Lambda_l$ satisfies the relation

$$\begin{aligned} U^l_R &:= E(u^l_{(R)}; \Omega_R \setminus \Lambda_l) - \int\limits_{\Gamma_R} g^l_{(R)} u^l_{(R)}\,ds_x = -\frac{1}{2}\int\limits_{\Gamma_R} g^l_{(R)} u^l_{(R)}\,ds_x \\ &= -\frac{1}{2}\int\limits_{\Gamma_R} g^l_{(R)}(x)\Big(u^l(x) + R^{-1/2}v\Big(\frac{x}{R}\Big)\Big)\,ds_x + O\Big(m_{11}C_1(l)^2\,\frac{l}{R}\Big). \end{aligned} \tag{3.7}$$

We consider the last integral I_R over the contour $\Gamma_R = \partial\Omega_R$. In view of formulas (2.4) and (2.5), we have

$$\begin{aligned} I_R &= RI^0 + \sum_{j=1}^{2} C_j(l) I^j_R, \qquad I^0 = \sum_{j,k=1}^{2} C_j(l)C_k(l)\int\limits_{\Gamma_1} \sigma^{(n)}(X^j; x)X^k(x)\,ds, \\ I^j_R &= \int\limits_{\Gamma_R} \sigma^{(n)}(X^j; x)\Big(u^l(x) - \sum_{k=1}^{2} C_k(l)X^k(x) + R^{-1/2}v\Big(\frac{x}{R}\Big)\Big)\,ds. \end{aligned} \tag{3.8}$$

The quantity I^0 does not depend on R. It should be noted that by virtue of relations (2.8) and (3.5), the difference $u^l - \sum C_k(l)X^k$ coincides with accuracy up to $O(m_{11}C_1(l)(L/R)^{3/2})$ with the expression $\sum N_k(l)\, Y^k$ on the contour Γ_R and the following equality is valid:

$$\sigma^{(n)}\Big(\sum_{k=1}^{2} N_k(l)Y^k(x) + R^{-1/2}v(R^{-1}x)\Big) = 0, \qquad x \in \Gamma_R.$$

Hence,

$$I_R^j = q\Big(X^j, \sum_{k=1}^{2} N_k(l)Y^k + R^{-1/2}v; \Gamma_R\Big) + O\Big(m_{11}C_1(l)^2\,\frac{l}{R}\Big)$$

$$= \sum_{k=1}^{2} N_k(l)q(X^j, Y^k; \Gamma) + O\Big(m_{11}C_1(l)^2\,\frac{l}{R}\Big) = N_j(l) + O\Big(m_{11}C_1(l)^2\,\frac{l}{R}\Big). \tag{3.9}$$

Thus, from formulas (3.7)–(3.9) it follows that the potential energy U_R^l increases without bounds as $R \to +\infty$, but the increment $\Delta U_R^l = U_R^l - U_R^0$ remains bounded and tends to the quantity

$$\Delta U^l = -\frac{1}{2}\sum_{j=1}^{2} C_j(l)N_j(l), \tag{3.10}$$

which should be understood as the increment in the potential strain energy of an unbounded solid due to crack extension.

4. Asymptotic Representation of the Total-Energy Increment. The surface-energy increment for the formation of the branch (2.1) of the crack $\Lambda_l \setminus \Lambda_0$ is easy to find:

$$\Delta S^l = S^l - S^0 = 2\int_0^l \gamma\Big(\arctan\Big(h\,\frac{dH}{dx_1}(x_1)\Big)\Big)\Big(1 + h^2\Big|\frac{dH}{dx_1}(x_1)\Big|^2\Big)^{1/2} dx_1$$

$$= 2\gamma(0)l + h^2(\gamma(0) + h^2\gamma''(0))\int_0^l \Big|\frac{dH}{dx_1}(x_1)\Big|^2 dx_1 + O(h^4). \tag{4.1}$$

Here γ is the surface-energy density, which smoothly depends on the crack inclination angle $\theta = \arctan\,(h \times dH(x_1)/dx_1)$; the prime denotes the derivative with respect to θ; $\gamma'(0) = 0$, since γ is an even function of the variable θ by virtue of the assumption of the symmetry of the physical properties of the material about the abscissa.

According to [11], the increment in the potential strain energy (3.10) is calculated by the formula

$$\Delta U^l = -\frac{1}{2}\sum_{j,k=1}^{2} C_j(l)M_{jk}(l, hH)C_k(l), \tag{4.2}$$

where $M = (M_{jk})$ is the symmetric positive definite matrix of energy release of size 2×2, which is composed of the coefficients of the expansion

$$w^j(x) = X^j(x) + \sum_{k=1}^{2} M_{jk}(l, hH)Y^k(x) + O(|x|^{-1}) \qquad (|x| \to \infty) \tag{4.3}$$

of the special solutions of the problem (2.2), (2.3) of a plane with a crack Λ_l. By virtue of relations (2.4) and (4.3), the equality $u^l = C_1(l)w^1 + C_2(l)w^2$ holds, which, together with relations (2.8) and (3.10), leads to formula (4.2).

To obtain an asymptotic representation (with respect to the parameter h) of the coefficients $M_{jk}(l, hH)$, we use the boundary rectifying method (for an alternative approach see [15, 16]), which is used in [2, 12–14] and substantiated in [7]. Performing the change

$$x = (x_1, x_2) \ \mapsto\ \xi = (\xi_1, \xi_2) = (x_1 - l, x_2 - hH(l)),$$

we place the origin of the Cartesian coordinates ξ at the tip of the crack (2.1), which in these coordinates is defined by the formula

$$\Lambda_l := \Lambda_l(h, H) = \{\xi\colon \xi_1 \le 0,\ \xi_2 = h(H(\xi_1 + l) - H(l))\}. \tag{4.4}$$

Because for the positively homogeneous power $\lambda \in \mathbb{R}$ of the function Z for any constant vector $b \in \mathbb{R}^2$, the Taylor formula

$$Z(\xi + b) = Z(\xi) + (b\nabla_\xi)Z(\xi) + O(|\xi|^{\lambda-2}) \qquad (|\xi| \to \infty)$$

is valid, the asymptotic condition (4.3) for the vector function $\xi \mapsto W^j(\xi) = w^j(x)$ is written as

$$W^j(\xi) = X^j(\xi) + l\,\frac{\partial X^j}{\partial \xi_1}(\xi) + hH(l)\,\frac{\partial X^i}{\partial \xi_2}(\xi) + \sum_{k=1}^{2} M_{jk}(l, hH)Y^k(\xi) + O(|\xi|^{-1}), \\ |\xi| \to \infty. \tag{4.5}$$

Seeking an asymptotic representation for the parameter h of the fields

$$W^j(\xi) = W^{j0}(\xi) + hW^{j1}(\xi) + h^2 W^{j2}(\xi) + \dots, \tag{4.6}$$

we extend the boundary conditions from the crack (4.4) onto the half-line $\Lambda_0 = \{\xi\colon \xi_1 \le 0,\ \xi_2 = 0\}$. The outward normal $n^\pm(h, \xi_1)$ to the faces $\Lambda_l^\pm$ is specified by the formula

$$n^\pm(h, \xi_1) = n_0(h, \xi_1)^{-1/2}(\pm hH_l'(\xi_1), \mp 1), \tag{4.7}$$

where $H_l(\xi_1) = H(\xi_1 + l) - H(l)$ and $n_0(h, \xi_1) = 1 + h^2 H_l'(\xi_1)^2$; the prime denotes the derivative with respect to ξ_1. Substituting relations (4.6) and (4.7) into the boundary condition (2.3) multiplied by $n_0(h, \xi_1)^{1/2}$, we formally obtain the relation

$$\begin{aligned} 0 = \mp(\sigma_{2k}(W^j; h, \xi) - hH_l'(\xi_1)\sigma_{1k}(W^j; h, \xi))\Big|_{\xi_2 = hH_l(\xi_1)} = \mp\sigma_{2k}(W^{j0}; \xi_1, \pm 0) \mp \\ \mp h(\sigma_{2k}(W^{j1}; \xi_1, \pm 0) + H_l(\xi_1)\partial_2\sigma_{2k}(W^{j0}; \xi_1, \pm 0) - H_l'(\xi_1)\sigma_{1k}(W^{j0}; \xi_1, \pm 0)) \\ \mp h^2(\sigma_{2k}(W^{j2}; \xi_1, \pm 0) + H_l(\xi_1)\partial_2\sigma_{2k}(W^{j1}; \xi_1, \pm 0) - H_l'(\xi_1)\sigma_{1k}(W^{j1}; \xi_1, 0) \\ + (1/2)H_l(\xi_1)^2\partial^2\sigma_{2k}(W^{j0}; \xi_1, \pm 0) - H_l'(\xi_1)H_l(\xi_1)\partial_2\sigma_{1k}(W^{j0}; \xi_1, \pm 0)) + \dots\ . \end{aligned} \tag{4.8}$$

Each of the terms W^{jp} in the asymptotic relation (4.6) satisfies the homogeneous equilibrium equations (2.2) on the plane with a slit $\mathbb{R}^2 \setminus \Lambda_0$. The equalities $\partial_2\sigma_{2k}(W^{jp}) = -\partial_1\sigma_{1k}(W^{jp})$ implied by system (2.2) simplify the factors at h^p on the right side of (4.8). Cancelling these factors, we obtain the following boundary conditions on the faces of the crack Λ_0:

$$\sigma_{2k}(W^{j0}; \xi_1, \pm 0) = 0, \qquad k = 1, 2, \quad \xi_1 < 0; \tag{4.9}$$

$$\sigma_{2k}(W^{j1}; \xi_1, \pm 0) = \frac{\partial}{\partial \xi_1}\,(H_l(\xi_1)\sigma_{1k}(W^{j1}; \xi_1, \pm 0)), \qquad k = 1, 2, \quad \xi_1 < 0; \tag{4.10}$$

$$\begin{aligned} \sigma_{2k}(W^{j2}; \xi_1, \pm 0) = \frac{\partial}{\partial \xi_1}\,(H_l(\xi_1)\sigma_{1k}(W^{j1}; \xi_1, \pm 0)) \\ + \frac{1}{2}\,\frac{\partial}{\partial \xi_1}\Big(H_l(\xi_1)^2\frac{\partial}{\partial \xi_2}\,\sigma_{1k}(W^{j0}; \xi_1, \pm 0)\Big), \qquad k = 1, 2, \quad \xi_1 < 0. \end{aligned} \tag{4.11}$$

Relation (4.8) uses the Taylor formula for the variable ξ_2, which requires sufficient smoothness of the vector functions W^j and W^{jp}. The initial crack (4.4) has angular points $\xi^0 = (0, 0)$ and $\xi^{h\pm} = (-l, hH(0) \pm 0)$, at which the stresses $\sigma_{ik}(W^j; h, \xi)$ acquire singularities. In the straightening of the crack Λ_l, its tip ξ^0 remains motionless; therefore, the stresses $\sigma_{ik}(W^{jp}; \xi)$ possess the same singularities $O(|\xi|^{-1/2})$ as the stresses $\sigma_{ik}(W^j; h, \xi)$. On the right sides of (4.10) and (4.11), the higher-order singularities that arise from the differentiation of the stresses are cancelled by the factors $H_l(\xi_1)$, which vanish for $\xi_1 = 0$.

The point $\xi^{0\pm} = (-l, \pm 0) \in \Lambda_0^{\pm}$ is the image of the vertex $\xi^{h\pm}$ of an angle with the opening $\pi \mp \arctan\,(hH'(0))$ on $\Lambda_l^{\pm}$, and the stresses $\sigma_{ik}(W^j; h, \xi)$ are quantities of order $O(|\xi - \xi^{h\pm}|^{\beta_\pm(h)})$ near the points $\xi^{h\pm}$ [$\beta_\pm(h)$ are infinitesimal as $h \to +0$] (see, for example, [17]). By virtue of the results obtained in [7] (see also [18, 19]), the expansion (4.6) can be imparted meaning only provided that the order λ of the singularity $\sigma_{ik}(W^{jp}; \xi) = O(|\xi - \xi^{0\pm}|^{-\lambda})$ it is strictly less than unity. Otherwise, additional boundary layers arise near the points $\xi^{h\pm}$; they are absent in the representation (4.6) but are constructed using the procedures described in [7, 17–19]. Since the function H_l is smooth everywhere except at the point $\xi_1 = -l$, the right sides of boundary conditions (4.10) can undergo discontinuities of the first kind at this point, and the stresses $\sigma_{ik}(W^{j1})$ acquire a logarithmic singularity $O(1 + |\ln|\xi - \xi^{0\pm}||)$. In this case, the right sides of boundary conditions (4.11) and, hence, the stresses $\sigma_{ik}(W^{j2}; \xi)$ have nonenergy singularities $O(|\xi - \xi^{0\pm}|^{-1})$ and, as noted above, they require a correction near the points $\xi^{h\pm}$. Thus, in the case of discontinuities and ignoring the boundary layers, one needs to eliminate the singular term $h^2 W^{j2}(\xi)$ from expansion (4.6), i.e., to confine oneself to two terms of the asymptotic representation and thus lower the accuracy of the expansion. We note that according to [20], the differentiation of the specified piecewise smooth boundary conditions leads to the formation of Dirac δ-functions, which do not appear explicitly in (4.11). From an analysis of the behavior of the solution $W^{j2}(\xi)$ as $\xi \to \xi^{0\pm}$, it follows that the accuracy $o(h^2)$ of expansion (4.6) is retained outside the fixed vicinity of the point $\xi = (-l, 0)$, i.e., the boundary layer near the vertices $\xi^{h\pm}$ of "almost straight" angles only has an indirect influence on the terms of the asymptotic representation

$$M_{jk}(l, hH) = M^0_{jk} + hM^1_{jk} + h^2 M^2_{jk} + \ldots, \tag{4.12}$$

which are the coefficients in expansion (4.5) at infinity. Nevertheless, the accuracy of the asymptotic constructions ignoring the boundary layers is acceptable for the attainment of the basic goal of the present work.

We determine the terms W^{jp}. Substituting (4.6) and (4.12) into relation (4.5), by virtue of (2.10), we obtain

$$W^{j0}(\xi) = X^j(\xi) - lm_{jj} Y^j(\xi) + \sum_{k=1}^{2} M^0_{jk} Y^k(\xi) + O(|\xi|^{-1}). \tag{4.13}$$

It is obvious that the solution of the homogeneous equilibrium equations in $\mathbb{R}^2 \backslash \Lambda_0$ with the boundary and asymptotic conditions (4.9) and (4.13) has the form $W^{j0}(\xi) = X^j(\xi)$; therefore, the following equalities are valid:

$$M^0_{jj} = lm_{jj}, \quad j = 1, 2, \qquad M^0_{12} = M^0_{21} = 0. \tag{4.14}$$

For the next term W^{j1}, relation (4.5) implies the asymptotic representation

$$W^{j1}(\xi) = H(l)\,\frac{\partial X^j}{\partial \xi_2}(\xi) + \sum_{k=1}^{2} M^1_{jk} Y^k(\xi) + O(|\xi|^{-1}). \tag{4.15}$$

Boundary conditions (4.10) for $k = 2$ are homogeneous (have zero right sides) since X^j is a solution of the homogeneous elastic problem on a plane with a semi-infinite slit and, hence, $\sigma_{12}(X^j; \xi_1, \pm 0) = 0$, $\xi_1 < 0$. In [2], it is verified that

$$\sigma_{11}(X^1; \xi_1, \pm 0) = 0, \qquad \sigma_{11}(X^2; \xi_1, \pm 0) = \pm \sigma^0_{11} r^{-1/2} \neq 0, \qquad \xi_1 < 0. \tag{4.16}$$

Thus, in the case of $j = 1$, both boundary conditions (4.10) are homogeneous. By virtue of formula (2.11), relation (4.15) is written as

$$W^{11}(\xi) = m_{11} H(l) Y^2(\xi) + \sum_{k=1}^{2} M^1_{1k} Y^k(\xi) + O(|\xi|^{-1}) = O(|\xi|^{-1/2}). \tag{4.17}$$

The solution of the homogeneous problem of elasticity theory that vanishes at infinity is trivial, i.e., $W^{11}(\xi) = 0$. By virtue of the expansion (4.17), we obtain

$$M^1_{11} = 0, \qquad M^1_{12} = M^1_{21} = -m_{11} H(l). \tag{4.18}$$

From the second formula in (4.16), it follows that in the case $H_l'(-l) \neq 0$ for $j = 2$ and $k = 1$, the right part of boundary condition (4.10) undergoes a discontinuity at the point $\xi_1 = -l$; therefore, the construction of the term W^{22} is complicated. As in [2], it can be verified that $M^1_{22} = 0$ but this equality is not required below.

It remains to consider the term W^{12}. From formulas (4.5), (4.11), and (2.10) and relations $W^{10} = X^1$ and $W^{11} = 0$, we obtain

$$W^{12}(\xi) = \sum_{k=1}^{2} M_{1k}^2 Y^k(\xi) + O(|\xi|^{-1}), \qquad |\xi| \to \infty,$$

$$\sigma_{22}(W^{12}; \xi_1, \pm 0) = 0,$$

$$\sigma_{21}(W^{12}; \xi_1, \pm 0) = \frac{1}{2} \frac{\partial}{\partial \xi_1} \Big(H_l(\xi_1)^2 \sigma_{11}\Big(\frac{\partial X^1}{\partial \xi_2}; \xi_1, \pm 0\Big)\Big)$$

$$= -\frac{1}{2} \frac{m_{11}}{m_{22}} \frac{\partial}{\partial \xi_1} \Big(H_l(\xi_1)^2 \frac{\partial}{\partial \xi_1} \sigma_{11}(X^2; \xi_1, \pm 0)\Big), \qquad \xi_1 < 0.$$

We repeat the calculations that led to (3.9), using the Green formula in the region $\Omega_R \setminus \Lambda_0$. Then, taking into account the normalization (2.9), we obtain

$$M_{1k}^2 = q(X^2, W^{12}; \Gamma_R) = \lim_{R\to\infty} \sum_{\pm} \int_{-R}^{0} X_2^k(\xi_1, \pm 0)\, \sigma_{21}(W^{12}; \xi_1, \pm 0)\, d\xi_1 = 0,$$

$$k = 1, 2. \tag{4.19}$$

Equality (4.19) is obtained using the relations similar to formulas (4.16) that are proved in [2]:

$$X_2^1(\xi_1, \pm 0) = 0, \qquad X_2^2(\xi_1, \pm 0) = \pm X_2^{20} r^{1/2} \neq 0, \qquad \xi_1 < 0.$$

From the results of [7] and the analysis of the terms W^{jp} of the asymptotic representation (4.6), it follows that the error in determining the coefficients $M_{11}(l, hH)$ and $M_{12}(l, hH) = M_{21}(l, hH)$ of expansions (4.5) is $O(h^3)$ and the error in determining the coefficient $M_{22}(l, hH)$ is $O(h^2(1 + |\ln h|))$.

5. Use of the Griffith Criterion. According to formulas (2.7), (3.10), (4.1), (4.12), (4.14), (4.18), and (4.19), we have

$$\Delta T^l = \Delta U^l + \Delta S^l = 2\Big(\gamma(0) - \frac{1}{4} m_{11} C_1(l)^2 - \frac{h^2}{4} m_{22} \overline{C}_2(l)^2\Big) l$$

$$+ h^2 \Big((\gamma(0) + \gamma''(0)) \int_0^l \Big| \frac{dH}{dx_1}(x_1) \Big|^2 dx_1 + m_{11} H(l) C_1(l) \overline{C}_2(l) \Big) + O(h^3). \tag{5.1}$$

For the global formulation of the energy criterion of fracture, it is necessary to minimize expression (5.1). All parameters, except for the function H describing the crack shape (2.1), are known. The Euler equation for the functional

$$(\gamma(0) + \gamma''(0)) \int_0^l \Big| \frac{dH}{dx_1}(x_1) \Big|^2 dx_1 + m_{11} H(l) C_1(l) \overline{C}_2(l) \tag{5.2}$$

on the set of functions H from the Sobolev class $W_2^1(0, l)$ that satisfy the condition

$$H(0) = 0, \tag{5.3}$$

imply the differential equalities

$$-\frac{d^2 H}{dx_1^2}(x_1) = 0, \quad x_1 \in (0, l), \qquad \frac{dH}{dx_1}(l) = -\frac{1}{2} (\gamma(0) + \gamma''(0))^{-1} m_{11} C_1(l) \overline{C}_2(l). \tag{5.4}$$

The solution of the mixed boundary-value problem (5.4), (5.3) is obvious:

$$H(x_1) = -(1/2)(\gamma(0) + \gamma''(0))^{-1} m_{11} C_1(l) \overline{C}_2(l) x_1. \tag{5.5}$$

However, the physical interpretation of the result is complicated since the crack shape (2.1), (5.5) is determined only by the SIF $C_i(l)$ and ignores the loading prehistory. In particular, in the case $C_1(\lambda)C_2(\lambda) \neq C_1(l)C_2(l)$ for

$\lambda < l$, the crack Λ_λ does not lie on the crack Λ_l because the branch $\Lambda_\lambda \setminus \Lambda_0$ is directed to the abscissa at an angle different from the angle at which the branch $\Lambda_l \setminus \Lambda_0$ is directed. At the same time, the crack cannot change path since the fracture process is irreversible. This contradiction implies that with the global formulation of the energy criterion of fracture that leads to the problem of the minimum of the quadratic functional (5.2) is not adequate to the crack curving process under a variable load.

We consider the local Griffith criterion. As the parameter τ, we use the increment l of the length of the crack projection onto the abscissa. We compare the positions of the crack at the times $\tau = l$ and $\tau + \Delta\tau = l + \Delta l$ ($l^{-1}\Delta l$ is a small positive number). By virtue of formula (5.1), which is written for l and $l + \Delta l$, the total-energy increment on the interval $(\tau, \tau + \Delta\tau)$ is equal to

$$\Delta T = 2\Big(\gamma(0) - \frac{1}{4}\,(m_{11}C_1(l)^2 - h^2 m_{22}\overline{C}_2(l)^2) - \frac{l}{2}\,\frac{d}{dl}\,(m_{11}\,C_1(l)^2 - h^2 m_{22}\overline{C}_2(l)^2)\Big)\Delta l$$

$$+h^2\Big((\gamma(0)+\gamma''(0))\Big|\frac{dH}{dx_1}(l)\Big|^2 + m_{11}\,\frac{dH}{dx_1}(l)\,C_1(l)\overline{C}_2(l) + m_{11}H(l)\,\frac{d}{dl}\,(C_1(l)\,\overline{C}_2(l))\Big)\Delta l + O(h^3 + \Delta l^2). \tag{5.6}$$

Expression (5.6) depends on the material parameters m_{11}, m_{22}, and $\gamma(0)$, the SIFs $C_i(l)$, the rates of their change, and the quantities $H(l)$ and $(dH/dx_1)(l)$, which characterize the position of the crack tip and the angle of its deviation from the abscissa [compare with the definition (2.1)]. At the time $\tau = l$, all data of the problem, except for the derivative $(dH/dx_1)(l)$, are fixed and the minimum of the quadratic function

$$\frac{dH}{dx_1}(l) \mapsto (\gamma(0)+\gamma''(0))\,\Big|\frac{dH}{dx_1}(l)\Big|^2 + m_{11}\,\frac{dH}{dx_1}(l)\,C_1(l)\overline{C}_2(l)$$

is reached for

$$\frac{dH}{dx_1}(l) = -\frac{1}{2}\,(\gamma(0)+\gamma''(0))^{-1} m_{11} C_1(l)\overline{C}_2(l). \tag{5.7}$$

The solution of the Cauchy problem (5.7), (5.3) is given by the equality

$$H(x_1) = -\frac{1}{2}\,(\gamma(0)+\gamma''(0))^{-1} m_{11} \int_0^{x_1} C_1(l)\overline{C}_2(l)\,dl. \tag{5.8}$$

The crack shape (2.1), (5.8) depends on the entire loading history. After elimination of the parameter h from formulas (2.1), (2.7), and (5.8), the crack path equation becomes

$$x_2 = -\frac{1}{2}\,(\gamma(0)+\gamma''(0))^{-1} m_{11} \int_0^{x_1} C_1(l)C_2(l)\,dl \tag{5.9}$$

under the assumption of smallness of the SIF ratio $C_2(l)/C_1(l)$. If $C_2(l) = 0$ for $l \in (0, l^0)$, the crack Λ_{l^0} remains straight.

Substituting formulas (5.9) and (2.7) into relation (5.6), we find the rate of release of the total energy during growth of a curved crack:

$$-\frac{dT}{dl}(l) = \frac{1}{2}\,\frac{d}{dl}\,\Big(l(m_{11}C_1(l)^2 + m_{22}C_2(l)^2)\Big) - 2\gamma(0)$$

$$+\frac{1}{4}\,\frac{m_{11}^2}{\gamma(0)+\gamma''(0)}\Big\{C_1(l)^2C_2(l)^2 + 2\int_0^l C_1(\lambda)C_2(\lambda)\,d\lambda\,\frac{d}{dl}\,(C_1(l)C_2(l))\Big\} + O(h^3). \tag{5.10}$$

For hypothetically straight-line crack propagation under the action of a mixed load, the rate of release of the total energy is calculated without constraints on the second-mode SIF:

$$-\frac{dT}{dl}(l) = -\frac{dS}{dl}(l) - \frac{\partial U}{\partial l}(l) = \frac{1}{2}\,\frac{d}{dl}\,\Big(l(m_{11}C_1(l)^2 + m_{22}C_2(l)^2)\Big) - 2\gamma(0). \tag{5.11}$$

Formula (5.11) coincides with the classical Griffith formula for constant SIFs $C_i(l)$. Expression (5.10) contains an additional term of the next order of smallness compared to $m_{11}C_1(l)^2$. This term is due to crack curving and depends on the rate of load variation (the derivative with respect to l) and its prehistory (the integral over λ). Both dependences are easily predicted. First, the potential strain-energy increment is due not only to the crack growth but also to the load variation (see, for example, [21]). Second, formulas (5.1) and (5.6) contain the total deviation $hH(l)$ of the tip of the crack Λ_l from the abscissa, which is determined by the entire loading process.

If the SIF $C_2(\lambda)$ is not equal identically to zero on the interval $(0, l)$, then the expression in braces in (5.10) is not necessarily positive. Thus, the rate of energy release (5.11) can exceed the rate (5.10); this, however does not lead to straight-line propagation of the crack because it is already curved, and formula (5.11) cannot be used.

We consider the factor $(\gamma(0)+\gamma''(0))^{-1}$ in relations (5.8)–(5.10), which reflects the anisotropy of the strength properties of the material. If the direction $\theta = 0$ corresponds to the minimum of the surface energy density $\gamma(\theta)$ [it was assumed that $\gamma'(0) = 0$], $\gamma''(0) > 0$ and, hence, in comparison with the case of strength isotropy, the path is flattened and the critical load increases. If $\theta = 0$ is the maximum point of the density γ, the curvature of the plot increases and the critical load decreases. In addition, formulas (5.8)–(5.10) become meaningless for $\gamma''(0) = -\gamma(0)$ [for example, $\gamma(\theta) = \cos\theta + O(|\theta|^3)$]. In other words, a rapid decrease in the surface energy with increasing angle $|\theta|$ causes a sharp deviation of the crack from the abscissa, and the asymptotic analysis performed in Sec. 4, becomes invalid.

From the assumption, adopted in Sec. 2, that at any time τ the load is critical and the crack is at equilibrium, it follows implies that $dT/dl = 0$ for all $l \geq 0$. According to formula (5.10), this condition imposes a constraint on the SIFs. One of the possible situations is constant critical SIFs $C_1(l) = C_1(0)$ and $C_2(l) = C_2(0)$, for which the asymptotic model of the Griffith criterion predicts crack propagation along a half-line at the angle $-\arctan((\gamma(0) + \gamma''(0))^{-1} m_{11} C_1(0) C_2(0)/2)$ to its initial direction. If the second-mode SIF $C_2(l) = C_2(0) \neq 0$ is constant, the first-mode SIF $C_1(l)$ is found from the Cauchy problem for the nonlinear ordinary differential equation

$$Y''(l)Y(l) + Y'(l)^2/2 + m_{11}^{-1}(\gamma(0) + \gamma''(0))C_2(0)^{-2}(Y'(l)^2 + 2lY''(l)Y'(l))$$

$$= m_{11}^{-2}(4\gamma(0)C_2(0)^{-2} - m_{22}) \quad [l > 0, \qquad Y(0) = 0]$$

for the unknown

$$Y(l) = \int_0^l C_1(\lambda)\, d\lambda.$$

The SIF can be found from the last formula.

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